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Title

A FIXED POINT RESULT BY USING ALTERING DISTANCE FUNCTION

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Abstract:

In this paper we have proved a fixed point theorem in a complete metric space with the help of an altering distance function. The main feature of our theorem is that, it contains square root in the inequality. We have also deduced some consequences of our theorem and supported our result with examples.

Keywords: Contraction, Cauchy sequence, Fixed point, Altering distance function, Kannan type mapping.

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Introduction and Mathematical Preliminaries:

In this section we have given some existing definitions, results and notations which are essential for our discussion in the next sections.

Throughout this paper we have used the following notations.

We denote the set of real numbers by R and R^+ is the set of positive real numbers. If A and B are two non-empty sets, we define $A \setminus B = \{x : x \in A \text{ but } x \notin B\}$.

In 1922, Banach [3] proved a fixed point result involving some contraction. This contraction principle is one of the most important results in modern mathematics. After this a lot of fixed point results appeared in the literature. In 1984, Khan, Swaleh and Sessa [14] proved a new type of contraction mapping principle. They proved their result with the help of a control function, which they called altering distance function. Afterwards a number of works appeared in which altering distance function was used. We give below the definition of altering distance function:

Definition 1.1 ***Altering distance function [14]***

A function $h:[0,\infty) \to [0,\infty)$ *is an altering distance function if*

(i) h *is monotonic increasing and continuous and*

(ii) $h(t) = 0$, *if only if* $t = 0$.

Khan et al. proved the following generalization of Banach contraction mapping principle.

Theorem 1.1 ***[14]*** *Let* (X,d) *be a complete metric space,* h *be an altering distance function and let* $f : X \to X$ *be a self mapping which satisfies the following inequality*

$$h(d(fx, fy)) \leq ch(d(x, y)),$$

for all $x, y \in X$ *and for some* $0 < c < 1$. *Then* f *has a unique fixed point.*

In fact Khan et al. proved a more general theorem (Theorem 2 in [14]) of which the above result is a corollary.

References [1, 2, 18, 20, 21] and [22] are some of the examples of fixed point results in which single valued altering distance functions have been used. In [4], altering distance functions have been generalized to a two variable function and in [5] a three variable generalization have been introduced and applied for obtaining fixed point results.

Kannan [12, 13] introduced another type of contraction mappings.

Definition 1.2 ***[12,13]*** *Let* (X,d) *be a complete metric space and* f *be a mapping on* X. *The mapping* f *is called a Kannan type mapping if there exists* $0 \leq \alpha < \frac{1}{2}$ *such that*

$$d(fx, fy) \leq \alpha[d(x, fx) + d(y, fy)], \text{ for all } x, y \in X .$$

Kannan type mappings are considered to be important in metric fixed point theory for several reasons. We mention two mathematical reasons in the following.

A mapping satisfying Banach contraction is continuous. A natural question is whether there exists a class of mappings satisfying some contractive inequality which necessarily have fixed points in complete metric spaces but need not necessarily be continuous. Kannan type mappings are such mappings to be first discovered [12, 13]. Another reason is its connection with metric completeness. A Banach contraction mapping may have a fixed point in metric space which is not complete. In fact, Connell, in [9], has given an example of a metric space which is not complete but every Banach contraction defined on this metric space has a fixed point. It has been established in [23], that the metric completeness is implied by the essential existence of fixed points of the class of Kannan type mappings.

There are a large number of works dealing with Kannan type mappings. Several examples of these works are noted in [15, 16] and [19].

B.S. Choudhury and K.P. Das introduced a probabilistic generalization of altering distance function in [6]. After this a lot of fixed point and coincidence point results appeared in probabilistic spaces. Some of the references are noted in [7, 8, 10, 11] and [17].

The purpose of this paper is to prove a fixed point result in complete metric spaces involving altering distance function. Our result is a generalization of the Banach contraction mapping principle and fixed point results due to Kannan. We have deduced some consequences from our theorem and our result is also supported by examples.

The Main Theorem:

Theorem 2.1 Let a,b,c,d *be four decreasing functions from* $R^+ \setminus \{0\}$ *into* $[0,1)$ *satisfy the property* $a(t)+b(t)+c(t)+4d(t)<1$, *for all* $t>0$. *Also let* (X,ρ) *be a complete metric space,* ψ *be an altering distance function and* $f: X \to X$ *be a self mapping which satisfies the following properties:*

$$\psi(\rho(fx,fy)) \le a(\rho(x,y)).\psi(\rho(x,y)) + b(\rho(x,y)).\psi(\rho(x,fx)) + c(\rho(x,y)).\psi(\rho(y,fy))$$

$$+ d(\rho(x,y)).\sqrt{\psi(\rho(x,fy))}.\sqrt{\psi(\rho(y,fx))} \tag{2.1}$$

and

$$\text{whenever } \lim_{n\to\infty} x_n = z\text{, then } \lim_{n\to\infty} fx_n = fz\text{,} \tag{2.2}$$

where $x,y \in X$ *and* $\{x_n\}$ *is a sequence of points from* X. *Then* f *has a unique fixed point.*

Proof: Let x_0 be a point of X. We define $x_{n+1} = fx_n$, $\tau_n = \rho(x_n, x_{n+1})$, for all integers $n \geq 0$.

We first prove that f has a fixed point. We may take $\tau_n > 0$, for all n, because if $\tau_n = 0$, there is a fixed point of f.

Substituting $x = x_n$ and $y = x_{n+1}$ in (2.1) we get

$$\psi(\tau_{n+1}) \leq a(\tau_n).\psi(\tau_n) + b(\tau_n).\psi(\tau_n) + c(\tau_n)\psi(\tau_{n+1})$$

$$+ d(\tau_n).\sqrt{\psi(\rho(x_n, x_{n+1}))}.\sqrt{\psi(\rho(x_{n+1}, x_{n+1}))}. \tag{2.3}$$

Hence we get

$$\psi(\tau_{n+1}) \leq \frac{a(\tau_n) + b(\tau_n)}{1 - c(\tau_n)}.\psi(\tau_n) < \psi(\tau_n), \tag{2.4}$$

because $a(t) + b(t) + c(t) + 4d(t) < 1$, implies $\frac{a(t) + b(t)}{1 - c(t)} < 1$, for all $t > 0$.

Since ψ is an increasing function we get from (2.4), $\{\tau_n\}$ is a decreasing sequence which is bounded below.

$$\text{Let } \lim_{n \to \infty} \tau_n = \tau. \tag{2.5}$$

We clam that $\tau = 0$. If possible, let $\tau > 0$. Then by (2.4) we get $\tau_n \geq \tau$, which implies that

$$\psi(\tau_{n+1}) \leq \frac{a(\tau) + b(\tau)}{1 - c(\tau)}.\psi(\tau_n). \tag{2.6}$$

Letting $n \to \infty$ in (2.6), since ψ is continuous, we obtain

$$\psi(\tau) \le \frac{a(\tau)+b(\tau)}{1-c(\tau)}.\psi(\tau) < \psi(\tau), \tag{2.7}$$

which is impossible. Hence $\tau = 0$.

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \ge n$ such that

$$\rho(x_{m(k)}, x_{n(k)}) \ge \varepsilon. \tag{2.8}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfies (2.8). Then

$$\rho(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.9}$$

Let $s_n = \rho(x_{n(k)}, x_{m(k)})$, for all $n \ge 0$. Then we get

$$\varepsilon \le s_n \le \rho(x_{n(k)-1}, x_{n(k)}) + \rho(x_{n(k)-1}, x_{m(k)}) < \tau_{n(k)-1} + \varepsilon. \tag{2.10}$$

Letting $k \to \infty$ in (2.10) we get,

$$\text{since } \lim_{n\to\infty} \tau_n = 0 \text{ then } \lim_{n\to\infty} s_n = \varepsilon. \tag{2.11}$$

Note that $k \to \infty$ implies $n \to \infty$.

Also from triangle inequality we get, for all $n \ge 0$,

$$-\tau_{n(k)} - \tau_{m(k)} + s_n \le \rho(x_{n(k)+1}, x_{m(k)+1}) \le \tau_{n(k)} + \tau_{m(k)} + s_n. \tag{2.12}$$

Letting $k \to \infty$ in (2.12) we get

$$\lim_{n\to\infty}\rho(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon . \tag{2.13}$$

Substituting $x = x_{n(k)}$ and $y = x_{m(k)}$ in (2.1) we get

$$\psi(\rho(x_{n(k)+1}, x_{m(k)+1})) \le a(s_n).\psi(s_n) + b(s_n).\psi(\tau_{n(k)}) + c(s_n).\psi(\tau_{m(k)})$$

$$+ d(s_n).\sqrt{\psi(\rho(x_{n(k)}, x_{m(k)+1}))}.\sqrt{\psi(\rho(x_{m(k)}, x_{n(k)+1}))} . \tag{2.14}$$

Again applying triangle inequality in (2.14) we get

$$\psi(\rho(x_{n(k)+1}, x_{m(k)+1})) \le a(s_n).\psi(s_n) + b(s_n).\psi(\tau_{n(k)}) + c(s_n).\psi(\tau_{m(k)})$$

$$+ d(s_n).\sqrt{\psi(s_n + \tau_{m(k)})}.\sqrt{\psi(s_n + \tau_{n(k)})} . \tag{2.15}$$

Letting $k \to \infty$ in (2.15) we get

$$\psi(\varepsilon) \le \{a(\varepsilon) + d(\varepsilon)\}.\psi(\varepsilon) < \psi(\varepsilon), \tag{2.16}$$

which is absurd. Therefore $\{x_n\}$ is a Cauchy sequence and hence it is convergent in a complete metric space X .

$$\text{Let } \lim_{n\to\infty} x_n = z . \tag{2.17}$$

We now show that z is a fixed point of f . Since each $\tau_n > 0$, there is a subsequence $\{x_{h(n)}\}$ of $\{x_n\}$ such that $x_{h(n)} \ne z$, for each $n \ge 0$. Let $\alpha_n = \rho(z, x_n)$, for all $n \ge 0$.

Substituting $x = x_{h(n)}$ and $y = z$ in (2.1) we get

$$\psi(\rho(x_{h(n)+1}, fz)) \le a(\alpha_{h(n)}).\psi(\alpha_{h(n)}) + b(\alpha_{h(n)}).\psi(\tau_{h(n)}) + c(\alpha_{h(n)}).\psi(\rho(z, fz))$$

$$+d(\alpha_{h(n)}).\sqrt{\psi(\alpha_{h(n)+1})}.\sqrt{\psi(\rho(x_{h(n)}, fz))}. \quad (2.18)$$

Note that a,b,c are all less than 1 and d is less than $\frac{1}{4}$.

Hence by applying triangle inequality again in (2.18) we get

$$\psi(\rho(x_{h(n)+1}, fz)) < \psi(\alpha_{h(n)}) + \psi(\tau_{h(n)}) + \psi(\rho(z, fz))$$

$$+\frac{1}{4}\sqrt{\psi(\rho(z, fz) + \rho(fz, fx_{h(n)}))}.\sqrt{\psi(\alpha_{h(n)} + \rho(z, fz))}. \quad (2.19)$$

Letting $n \to \infty$ in (2.19) and using (2.2) we get

$$\limsup_{n\to\infty} \psi(\rho(x_{h(n)+1}, fz)) \leq \frac{1}{2}\psi(\rho(z, fz)). \quad (2.20)$$

On the other hand, the triangle inequality implies that

$$\rho(z, fz) \leq \alpha_{h(n)} + \tau_{h(n)} + \rho(x_{h(n)+1}, fz). \quad (2.21)$$

Since ψ is increasing, from (2.21) we obtain by letting $n \to \infty$

$$\psi(\rho(z, fz)) \leq \limsup_{n\to\infty} \psi(\rho(x_{h(n)+1}, fz)). \quad (2.22)$$

From (2.20) and (2.22) we get $\psi(\rho(z, fz)) = 0$, which implies $\rho(z, fz) = 0$. So z is a fixed point of f.

To prove the uniqueness of the fixed point, let us suppose that z_1 and z_2 be two fixed points of f. Then from (2.1) we get

$$\psi(\rho(z_1,z_2)) = \psi(\rho(fz_1, fz_2)) \leq \{a(\rho(z_1,z_2)) + d((\rho(z_1,z_2))\}\psi((\rho(z_1,z_2)) < \psi((\rho(z_1,z_2)),$$

a contradiction. Hence we must have $z_1 = z_2$.

This completes the proof.

Some Consequences and Examples:

If we assume $d = 0$ and $b = c$ in (2.1) we get the following theorem:

Theorem 3.1 Let a,b *be two decreasing functions from* $R^+ \setminus \{0\}$ *into* $[0,1)$ *which satisfy the property* $a(t) + 2b(t) < 1$, *for all* $t > 0$. *Also let* (X,ρ) *be a complete metric space,* ψ *be an altering distance function and* $f : X \to X$ *be a continuous self mapping which satisfies the following property:*

$$\psi(\rho(fx, fy)) \leq a(\rho(x,y))\psi(\rho(x,y)) + b(\rho(x,y)).\{\psi(\rho(x,fx)) + \psi(\rho(y,fy))\},$$

where $x, y \in X$ *and* $\{x_n\}$ *is a sequence of points from* X. *Then* f *has a unique fixed point.*

If we put $b = c$ in (2.1), we get the following theorem:

Theorem 3.2 Let a,b,c *be three decreasing functions from* $R^+ \setminus \{0\}$ *into* $[0,1)$ *which satisfy the property* $a(t) + 2b(t) + 4c(t) < 1$, *for all* $t > 0$. *Also let* (X,ρ) *be a complete metric space,* ψ *be an altering distance function and* $f : X \to X$ *be a continuous self mapping which satisfies the following property:*

$$\psi(\rho(fx, fy)) \leq a(\rho(x,y))\psi(\rho(x,y)) + b(\rho(x,y)).\{\psi(\rho(x,fx)) + \psi(\rho(y,fy))\}$$

$$+c(\rho(x,y)).\sqrt{\psi(\rho(x,fy))}.\sqrt{\psi(\rho(y,fx))},$$

where $x, y \in X$ *and* $\{x_n\}$ *is a sequence of points from* X *. Then* f *has a unique fixed point.*

We have proved below that Theorem 2.1 is an improvement of the Banach Contraction Principle.

Theorem 3.3 (*Banach Contraction Principle*) *Let* (X,ρ) *be a complete metric space and* $T: X \to X$ *be a mapping such that*

$$\rho(Tx,Ty) \le \alpha\rho(x,y), \tag{3.1}$$

for all $x, y \in X$ and $0 < \alpha < 1$. Then T has a unique fixed point in X .

Proof: We know that if a mapping T satisfies the condition (3.1) in Theorem 3.3, it is a continuous mapping. Hence condition (2.2) of Theorem 2.1 is obviously satisfied by these types of maps. Let $\psi(x) = x$, for all $x \in [0,\infty)$. We now put $b = c = d = 0$ and $a = \alpha$ in Theorem 2.1 and we get the Banach Contraction Principle.

Putting $a = 0$, $b = c = \alpha$ and $d = 0$ in Theorem 2.1 we get result due to Kannan [12, 13]. Note that at this time we also set $\psi(x) = x$, for all $x \in [0,\infty)$.

Here we have given two examples, one in discrete case and the other in continuous case, which support our main theorem, that is, Theorem 2.1.

Example 3.1 *Let* $X = \{1,2,3\}$ *and* ρ *be defined by* $\rho(2,1) = \rho(1,2) = \frac{1}{2}$, $\rho(3,2) = \rho(2,3) = 1$, $\rho(1,3) = \rho(3,1) = 1$ *and* $\rho(1,1) = \rho(2,2) = \rho(3,3) = 0$. *Then* (X,ρ) *is a complete metric space. We now take* $f: X \to X$ *be a function, defined by* $f1 = 2$, $f2 = 2$ *and* $f3 = 1$. *Also let* ψ *be* *an altering distance function defined by* $\psi(x) = x$, *for all* $x \in [0,\infty)$. *Then the function* f *satisfies all the conditions of Theorem 2.1 and we see that it has a unique fixed point 2 in* X .

Example 3.2 *Let* $X = [0,1]$ *and* $\rho(x, y) = |x - y|$, *for all* $x, y \in X$. *Then* (X,ρ) *is a complete metric space. Let* $f : X \to X$ *be defined by* $f(x) = \frac{x}{2}$, *for all* $x, y \in X$, *then obviously* f *is a continuous function. Also let* ψ *be an altering distance function defined by* $\psi(x) = x$, *for all* $x \in [0,\infty)$. *Then the function* f *satisfies all the conditions of Theorem 2.1 and we see that it has a unique fixed point 0 in* X .

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